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Time-of-arrival formalism for the relativistic particle

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Abstract. A suitable operator for the time-of-arrival at a detector is defined for the free relativistic particle in $(3+1)$ dimensions. For each detector position there exists a subspace of detected states in the Hilbert space of solutions to the Klein–Gordon equation. Orthogonality and completeness of the eigenfunctions of the time-of-arrival operator apply inside this subspace, opening up a standard probabilistic interpretation.

1. Introduction

In non-relativistic dynamics time has a characterization of its own which distinguishes it sharply from the space coordinates of configuration space. However, this difference can be simply removed at the formal level by going to the parametrized form of dynamics where time is made to depend on a parameter τ in as much as the coordinates q^i do. One is thus led to deal with a set $(q^i(\tau), t(\tau))$ in which the identification of time against coordinates appears more as a matter of convention than as a matter of significance from the point of view of the dynamical system under study. Even so time still maintains a particular role from the physical point of view. Time is experienced by the observer as well as by the system. This is more evident in the transition to quantum mechanics, where time—as opposed to position—cannot be viewed as a property of the system under scrutiny.

There is a way out of this situation as shown in [1], whose authors show how to deal with and solve the question *at what time* in quantum mechanics in one space dimension, by introducing a suitable time operator and obtaining the associated time representation. The outcome is the emergence of a $x \leftrightarrow t$ equivalence in quantum mechanics in much the same way as there is one in classical mechanics. The question *at what time* joins the question *at what position* as answerable not only experimentally, but also within the realm of the quantum mechanical formalism.

In special relativity time is obviously q^0, and it seems the question *at what time* would be addressed in relativistic quantum mechanics in a simple and direct way: explicit covariance should rule the presence of q^0 along with the space components q^i to form a Minkowski space four-vector q^μ. There should be no telling difference between the time and the space components of q, mainly taking into account that—in contrast with the non-relativistic case—they get entangled by Lorentz transformations. One could be led to believe in the existence of a spacetime position operator, a four-vector, whose components should transform covariantly under the Lorentz group. This object should address simultaneously

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the two questions *when* and *where* seemingly unrelated in the non-relativistic case. It is well known that this object has never been constructed. In the instant form of dynamics, i.e. referring the operators to their values at some instant of time, one can employ a three-vector operator—the position operator [2]—to answer the question *where*. This operator not only lacks explicit covariance, it also lacks a time component. The cause of these deficiencies can be traced back [3] to the reparametrization invariance of the action of the relativistic particle

$$\mathcal{S} = m \int \mathrm{d}\tau \sqrt{\dot{q}^2} \tag{1}$$

which translates into evolution (along τ) generated by a Hamiltonian $H = p^2 - m^2 = 0$. Since the Hamiltonian is constrained to vanish, the τ evolution is a gauge transformation. In the canonical approach one chooses a solution to the constraint, i.e. by putting $p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$, and 'fixes the gauge' by setting the evolution parameter to be the physical time. *A priori* there is no room left for the question *when* as there is no freedom left for a time operator differing from the time parameter q^0. This is a bonus from another point of view: demoting q^0 to the role of a parameter one evades the difficulty of a Hamiltonian unbounded from below in the same way as in the non-relativistic case. The lack of positivity of the density j^0 of the solutions of the Klein–Gordon equation also plays a role here. It brings about particle–antiparticle pairs, etc, and the untenability of the one-particle interpretation. From here on, the true variables are field configurations, to whom q^0, along with the space coordinates q^i, are mere parameters. However, the case of the relativistic particle we are analysing here is of intrinsic interest; it serves to set up the basis for the particle interpretation of quantum field theory, and also as a guideline to use [4] in the construction of the quantum formalism of the gravitational field. Analysing issues of time for the relativistic particle may prove valuable in transforming that formalism in a theory or, at least, may throw some light on the issues of time in quantum gravity [5]. This paper focuses on the relativistic particle. In section 2 we summarize the results of the canonical formalism, In section 3 we generalize the treatment of [1] to the free relativistic particle, section 4 contains the generalization to three space dimensions and section 5 is devoted to questions of orthogonality and completeness. Finally, in section 6 we discuss some issues raised by the interpretation of the formalism and some speculations about the applicability to quantum gravity.

2. Canonical formalism

Here we will focus our attention onto the physical Hilbert space $\mathcal{H}_{\mathrm{KG}}$ of the positive energy solutions $\psi(x)$ for the Klein–Gordon equation [6], with the understanding that negative energies will be reinterpreted in terms of antiparticles. In configuration space where the Klein–Gordon equation reads $(\Box + m^2)\psi(x) = 0$, the positive energy solutions are of the form:

$$\psi(x) = (2\pi)^{-3/2} \int \mathrm{d}^4k \, \mathrm{e}^{-\mathrm{i}kx} \delta(k^2 - m^2)\theta(k^0)\Psi(k) = (2\pi)^{-3/2} \int \frac{\mathrm{d}^3k}{2\omega(k)} \, \mathrm{e}^{-\mathrm{i}(\omega(k)x^0 - \boldsymbol{k}\boldsymbol{x})}\Psi(\boldsymbol{k}) \tag{2}$$

with a scalar product:

$$(\phi, \psi) = \mathrm{i} \int \mathrm{d}^3x \, (\phi^* \partial_t \psi - \partial_t \phi^* \psi) = \int \frac{\mathrm{d}^3k}{2\omega(k)} \Phi^*(\boldsymbol{k})\Psi(\boldsymbol{k}) \tag{3}$$

where $\omega(k) = \sqrt{\boldsymbol{k}^2 + m^2}$. We will follow the conventions of [7] denoting by uppercase letters the wavefunctions in momentum space, leaving the lower case for configuration space functions.

To answer the question 'What is the probability of finding the particle at the point $\boldsymbol{x}$ at time x^0?' with the above scalar product, we need to find a Hermitian position operator and find its eigenfunctions $\psi_{\boldsymbol{x},x^0}$. Then, the probability amplitude for finding a particle at $\boldsymbol{x}$ at time $x^0 = q^0$ is $(\psi_{\boldsymbol{x},x^0}, \phi)$, where $\phi(q)$ is the wavefunction giving the state of the particle. As shown by Newton and Wigner [2] the position operator is

$$\boldsymbol{Q} = \mathrm{i}\boldsymbol{\nabla}_p - \frac{\mathrm{i}\boldsymbol{p}}{2\omega^2(p)}. \tag{4}$$

In our notation, k will represent p in momentum space, while Qs and ps will denote operators, unless specified otherwise by the word 'classically', in which case they will denote classical dynamical variables. The eigenstate of the position operator localized at the point $\boldsymbol{x}$ at $t = 0$ is

$$\Psi_{\boldsymbol{x},0}(\boldsymbol{k}) = (2\pi)^{-3/2}\sqrt{2\omega(k)}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{x}}. \tag{5}$$

In general, given a particle in the state $\Phi(\boldsymbol{k})$ at $t = 0$, the probability amplitude to find it at the position $\boldsymbol{x}$ at $t = 0$ is given by

$$(\Psi_{\boldsymbol{x},0}, \phi) = (2\pi)^{-3/2}\int \frac{\mathrm{d}^3k}{2\omega(k)}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{x}}\sqrt{2\omega(k)}\Phi(\boldsymbol{k}). \tag{6}$$

The components of the position operator are in involution and their commutation relations with the momenta are canonical

$$[Q^i, Q^j] = 0 \qquad [Q^i, p^j] = \mathrm{i}\delta^{ij} \tag{7}$$

under rotations and space translations $\boldsymbol{Q}$ behaves as a three vector. It also evolves like the position of a particle should do, namely

$$\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t} = \mathrm{i}[\sqrt{\boldsymbol{p}^2 + m^2}, \boldsymbol{Q}] = \frac{\boldsymbol{p}}{\omega(p)}. \tag{8}$$

The Heisenberg position operator at time t can be obtained by integrating this equation

$$\boldsymbol{Q}(t) = \boldsymbol{Q} + \frac{\boldsymbol{p}}{\omega(p)}t. \tag{9}$$

We now would invert this equation to get an operator for the time-of-arrival of the relativistic particle following the proposal of [1]. Before doing this, we have some comments to make about the Newton–Wigner operator. This has some strange features that have been discussed in a vast literature to which we cannot do justice here. At least, it is worth recalling that, by construction, its eigenfunctions lack Lorentz covariance. Also, the localized state in configuration space $\psi_{\boldsymbol{x},0}(\boldsymbol{q}, 0)$ is not a delta function $\delta(\boldsymbol{x} - \boldsymbol{q})$ as in the non-relativistic case, but extends in space for $|\boldsymbol{x} - \boldsymbol{q}| \sim \Theta(\hbar/mc)$, and drops off exponentially only for large values of $|\boldsymbol{x} - \boldsymbol{q}|$. In addition, there is still some controversy about the very existence of a position operator in relativistic quantum mechanics [8]. In spite of all that, we have chosen this operator as the simplest one that being Hermitian, is associated with the variable $\boldsymbol{x}$ of configuration space, and possesses the canonical commutation relations of (7) and the correct time evolution (8). Given another position operator different from (4), the results of this paper could be applied to it by using the modified version of (4) in the appropriate places.

3. Time-of-arrival in one space dimension

The special role played by time has been the source of controversy since the early days of quantum mechanics. The search for the various time operators and the analysis of the associated time–energy uncertainty relations has been the subject of a number of works (see the bibliography in [9]), the outcome of which was that quantum mechanics cannot accommodate a time-of-arrival operator. This has been refuted recently in [9] where, in addition, an average value for this quantity is explicitly obtained for one space dimension in terms of the current density of the particle. This is framed in a wealth of recent works devoted to the issue of time in quantum mechanics—see [10] and the bibliography contained therein—with special emphasis on the tunnelling times, a question of fundamental and practical implications. Here, we are interested in the characterization of the time-of-arrival as one of the properties of the system under study as in [1], in other words, we need to go one step further and to obtain an associated operator to be able to analyse and give an interpretation of this property in the quantum formalism. This is necessary for our results to be of value for the quantum formalism of the gravitational field where, as said in the introduction, time has to be considered as a property of the system under study. For the sake of simplicity and also to connect with the non-relativistic one-dimensional case studied in [1] we begin by considering the case of one space dimension. Then we can rewrite (9) as

$$Q(t) = Q + \frac{p}{\omega(p)}t. \tag{10}$$

Classically, the time-of-arrival $t(X)$ at the position $Q(t) = X$ would be trivially given by inverting this equation

$$t(X) = (X - Q)\frac{\omega(p)}{p} \tag{11}$$

where Q is the initial position, X the detector position and p the particle momentum. In quantum mechanics Q and p do not commute, cf (7), and we have an operator ordering problem when trying to invert (10). What we can say is that the time-of-arrival operator could be given by the operator

$$Q^0(X) \simeq (X - Q)\frac{\omega(p)}{p} \tag{12}$$

where the symbol $\simeq$ is employed to mean equal up to ordering. In the non-relativistic case studied in [1], there was the same problem. There, the time-of-arrival operator $T(X)$ was given by

$$T(X) \simeq (X - Q)\frac{m}{p} \tag{13}$$

that was given the Hermitian solution

$$T(X) \simeq \mathrm{e}^{-\mathrm{i}px}\sqrt{\frac{m}{p}}(-Q)\sqrt{\frac{m}{p}}\,\mathrm{e}^{\mathrm{i}px} \tag{14}$$

with the non-relativistic position operator $Q = \mathrm{i}\partial/\partial p$. The relativistic case we are analysing brings about two modifications: m is replaced by $\omega(p)$ and $\mathrm{i}\partial/\partial p$ by the Newton–Wigner position operator (4). In this case we have

$$Q^0(X) = \mathrm{e}^{-\mathrm{i}pX}\sqrt{\frac{\omega(p)}{p}}\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}p} + \frac{\mathrm{i}p}{2\omega^2(p)}\right)\sqrt{\frac{\omega(p)}{p}}\,\mathrm{e}^{\mathrm{i}pX} \tag{15}$$

a form that goes to the operator $T(X)$ of [1] in the non-relativistic limit. The eigenfunctions of this operator

$$Q^0(X)\Psi_{T,X}(k) = T\Psi_{T,X}(k) \tag{16}$$

are given by

$$\Psi_{T,X}(k) = \alpha\sqrt{k}\,\mathrm{e}^{\mathrm{i}(\omega(k)T-kX)} \tag{17}$$

where α is a normalization factor. Multiplying by the phase factor $\exp(-\mathrm{i}mT)$, these functions give the eigenfunctions of [1] in the non-relativistic limit. We will not make distinctions between right- ($k > 0$) and left-moving ($k < 0$) particles here, as these have a meaning for one space dimension only and we want to study the three-dimensional (3D) case, where opposite directions can be connected continuously.

4. Three space dimensions

A new feature appears in three space dimensions that was not present in the case studied above. The space of 'detected' states is a subspace of the Hilbert space $\mathcal{H}_{\mathrm{KG}}$ of positive energy solutions to the Klein–Gordon equation. This comes about because in the 3D case the evolution equations that we have to invert to obtain the time-of-arrival is the set (9) of three equations depending on a unique parameter t. To be compatible, they have to satisfy the constraint

$$\boldsymbol{\mathcal{C}} = (\boldsymbol{Q} - \boldsymbol{X}) \wedge \boldsymbol{p} = 0 \tag{18}$$

where the 'point-of-arrival' $\boldsymbol{X}$ plays the role of a parameter and the symbol $\wedge$ indicates the exterior product. Classically, these constraints mean that the angular momentum of the particle is $\boldsymbol{X} \wedge \boldsymbol{p}$, so that $\boldsymbol{X}$ is a point in the particle trajectory, or simply that the angular momentum about $\boldsymbol{X}$ is zero. In quantum mechanics there are obstructions to imposing simultaneous values to different components of the angular momentum. At first sight, the best one can do is to constrain $\boldsymbol{L}^2$ and a component of the angular momentum, say L_3, to have definite values given from $\boldsymbol{X} \wedge \boldsymbol{p}$. However, this is not the case here, as we are equating the components of the angular momentum to an operator $\boldsymbol{X} \wedge \boldsymbol{p}$, in such a way that the constraints form a first-class system. Classically, (18) plays the role of a set of first-class constraints in the Hamiltonian formalism that we have to quantize following the method of Dirac. Now, the total Hamiltonian is

$$H = \sqrt{\boldsymbol{p}^2 + m^2} + \lambda_a \mathcal{C}_a \tag{19}$$

where

$$\mathcal{C}_a = \epsilon_{abc}(Q - X)_b p_c \tag{20}$$

and the ps and Qs are the dynamical variables to become operators after quantization. It is straightforward to show that

$$\{\mathcal{C}_a, \mathcal{C}_b\} = \epsilon_{abc}\mathcal{C}_c, \{\mathcal{C}_a, H\} = \epsilon_{abc}\lambda_b\mathcal{C}_c. \tag{21}$$

Therefore, we have a true first-class system, a different one for each vector $\boldsymbol{X}$.

There seems to be additional difficulties in that the eigenvalues of $\boldsymbol{L}^2$ and L_3 are integer numbers while the constraint will assign to them a continuous spectrum. Actually, this is not the case [11] because, even if the constraint can be written in the form $\boldsymbol{L} = \boldsymbol{X} \wedge \boldsymbol{p}$, this will not hold as an operator equation, nor the states on which it will be satisfied will be eigenstates of neither L_i nor $\boldsymbol{X}\wedge\boldsymbol{p}$. Now, the detected subspace $\mathcal{H}_{\mathrm{KG}}^{(X)}$ can be given simply as that spanned by the functions $\Psi^{(X)}(\boldsymbol{k})$ of the form

$$\mathcal{H}_{\mathrm{KG}}^{(X)} = \{\Psi^{(X)}(\boldsymbol{k}) = \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{X}}\Psi(k,\boldsymbol{X})\} \tag{22}$$

where $\Psi(k,\boldsymbol{X})$ represents an arbitrary function of the modulus of $\boldsymbol{k}$ and of $\boldsymbol{X}$. If we now require invariance under translations, we have to drop the dependence of $\Psi(k,\boldsymbol{X})$ on $\boldsymbol{X}$. In this case we can say that the Hilbert space $\mathcal{H}_{\mathrm{KG}}^{(X)}$ is obtained from $\mathcal{H}_{\mathrm{KG}}^{(0)}$ by a translation of amount $\boldsymbol{X}$.

We are now prepared to study $Q^0(\boldsymbol{X})$, the time-of-arrival at a point $\boldsymbol{X}$ in the 3D space. Classically, it is given by inverting the equation of motion:

$$Q^0(\boldsymbol{X}) = \frac{\omega(p)}{\boldsymbol{p}^2}(\boldsymbol{Q}-\boldsymbol{X})\cdot\boldsymbol{p} \tag{23}$$

which is a first-class dynamical variable $\{Q^0(\boldsymbol{X}),\mathcal{C}_a\} = 0$. In the Hilbert space $\mathcal{H}_{\mathrm{KG}}^{(X)}$ the operator equation of motion has to be rewritten with t replaced by the operator $Q^0(\boldsymbol{X})$ and $\boldsymbol{Q}(t)$ by the detector's position $\boldsymbol{X}$

$$\boldsymbol{X}-\boldsymbol{Q}-\frac{\boldsymbol{p}}{\omega(p)}Q^0(\boldsymbol{X}) = 0. \tag{24}$$

It should be an identity, with the operator Q^0 being such as to annihilate the left-hand side. By a vector product of the above equation by $\boldsymbol{p}$ we obtain the constraints that are already satisfied in the detected subspace. A scalar product by $\boldsymbol{p}$ gives

$$\boldsymbol{p}\boldsymbol{X}-\boldsymbol{p}\boldsymbol{Q}-\frac{p^2}{\omega(p)}Q^0(\boldsymbol{X}) = 0. \tag{25}$$

Putting

$$Q^0(\boldsymbol{X}) = \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\boldsymbol{X}}Q^0\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\boldsymbol{X}} \tag{26}$$

the previous equation reduces to

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}p}+\frac{\mathrm{i}p}{2\omega^2(p)}-\frac{p}{\omega(p)}Q^0 = 0. \tag{27}$$

Observe how, when acting on the detected subspace, (24) reduces effectively to only the one-dimensional equation (27). One would be tempted to solve it with the ordering chosen in (15), with eigenfunctions similar to (17). This choice would not do, as the norm of these states would be badly divergent in three-dimensional space. What we need are eigenstates with higher negative powers of k than in (17). This can be achieved by choosing a different ordering for the operator. Tentatively we put

$$Q^0 = \sqrt{\omega(p)}\frac{1}{p^{n+1}}\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}p}+\frac{\mathrm{i}p}{2\omega^2(p)}\right)p^n\sqrt{\omega(p)} \tag{28}$$

with this choice we get for the eigenfunction of (26) with eigenvalue T the expression

$$\Psi_T^{(X)}(\boldsymbol{k}) = \frac{1}{2\pi k^n}\,\mathrm{e}^{\mathrm{i}(\omega(k)T-\boldsymbol{k}\boldsymbol{X})} \tag{29}$$

where we have chosen some arbitrary fixed $\boldsymbol{X}$. We now choose n such that the scalar product be well behaved

$$(\psi_T^{(X)},\psi_{T'}^{X}) = \int\frac{\mathrm{d}^3k}{2\omega(k)}\Psi_T^{(X)*}(\boldsymbol{k})\Psi_{T'}^{(X)}(\boldsymbol{k}) = (2\pi)^{-1}\int_0^\infty \mathrm{d}k\,\frac{k^{2(1-n)}}{\omega(k)}\,\mathrm{e}^{\mathrm{i}\omega(k)(T'-T)}. \tag{30}$$

We see that the eigenfunctions are not orthogonal. We will address this problem in the next section. Now, we focus on the last integral, which strongly suggest the choice $n = 1/2$.

In the general case of d space dimensions we would chose $n = (d-2)/2$, to make the measure of the integral equal to $\mathrm{d}\omega$. Finally, in our case we have:

$$Q^0 = \sqrt{\omega(p)}\,p^{-3/2}\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}p} + \frac{\mathrm{i}p}{2\omega^2(p)}\right)p^{1/2}\sqrt{\omega(p)}$$
$$\Psi_T^{(X)}(\boldsymbol{k}) = (2\pi)^{-1}k^{-1/2}\,\mathrm{e}^{\mathrm{i}(\omega(k)T-\boldsymbol{k}\boldsymbol{X})} \tag{31}$$
$$(\psi_T^{(X)}, \psi_{T'}^{(X)}) = (2\pi)^{-1}\int_m^\infty \mathrm{d}\omega\,\mathrm{e}^{\mathrm{i}\omega(k)(T'-T)}.$$

If there is any doubt left in that the right choice is $n = 1/2$, one can check that this value gives the unique ordering that makes the operator Q^0 Hermitian, $(\phi, Q^0\psi) = (Q^0\phi, \psi)$.

5. Orthonormalization and completeness

The eigenfunctions of (31) are not yet orthogonal. However, the above scalar product is an appropriate expression for the Marolf's orthogonalization recipe [1]. It is based on the physical observation that for vanishing momentum the particle either never reaches the detector, or sits in it forever. To deal with this situation, Marolf proposed a regularization prescription for the time-of-arrival operator that 'avoids' zero-momentum particles. The procedure to follow is less obvious here than in the 1D non-relativistic case, due to the more complex structure of the operator. We first present the appropriate prescription for arbitrary n, returning to $n = 1/2$ at the end of the calculation, to show that only with this value does the procedure give orthogonal eigenfunctions in three space dimensions. First, we rewrite Q^0 in the momentum representation as

$$Q^0 = -\mathrm{i}\omega(k)\frac{1}{k^{n+1/2}\sqrt{k}}\frac{\mathrm{d}}{\mathrm{d}k}\frac{k^{n+1/2}}{\sqrt{k}} \tag{32}$$

which we regularize as follows

$$Q^0 = -\mathrm{i}\omega(k)\sqrt{f(k)}\frac{1}{k^{n+1/2}}\frac{\mathrm{d}}{\mathrm{d}k}k^{n+1/2}\sqrt{f(k)} \tag{33}$$

and where f is the same as in [1]

$$f(k) = \begin{cases} 1/k & \text{for } k > \epsilon \\ \epsilon^{-2}k & \text{for } k < \epsilon. \end{cases} \tag{34}$$

The eigenfunctions $\Psi_T^{(X)}(\boldsymbol{k})$ corresponding to this operator are of the form:

$$\Psi_T^{(X)}(\boldsymbol{k}) = \frac{1}{2\pi}\frac{\mathrm{e}^{\mathrm{i}(Z(k)T-\boldsymbol{k}\boldsymbol{X})}}{k^{n+1/2}\sqrt{f(k)}} \qquad Z(k) = \int_\epsilon^k \frac{\mathrm{d}k'}{\omega(k')f(k')} \tag{35}$$

and the orthogonality condition reads

$$(\psi_T^{(X)}, \psi_{T'}^{(X)}) = (2\pi)^{-2}\int \frac{\mathrm{d}^3k}{2\omega(k)f(k)}\frac{1}{k^{2n+1}}\,\mathrm{e}^{\mathrm{i}Z(k)(T'-T)}. \tag{36}$$

For the case $n = 1/2$ one gets

$$(\psi_T^{(X)}, \psi_{T'}^{(X)}) = (2\pi)^{-1}\int_{Z_{\min}}^{Z_{\max}} \mathrm{d}Z\,\mathrm{e}^{\mathrm{i}Z(T'-T)} = \delta(T-T') \tag{37}$$

as the coordinate Z goes from $-\infty$ to 0 as k goes from 0 to ϵ, and from 0 to ∞ as k goes from ϵ to ∞. Z and T form a pair of 'conjugate' variables in the subspaces $\mathcal{H}_{\mathrm{KG}}^{(X)}$. This can be seen from (37) and the associated completeness relation

$$\int_{-\infty}^{+\infty} \mathrm{d}T\, \Psi_T^{(X)}(\boldsymbol{k})\Psi_T^{(X)*}(\boldsymbol{k}') = \frac{1}{2\pi k^2 f(k)}\delta(Z(k) - Z(k'))\,\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{X}}. \tag{38}$$

The weird expression on the right-hand side is exactly what is needed to form a completeness relation in the detected subspace. For any function $\Phi^{(X)} \in \mathcal{H}_{\mathrm{KG}}^{(X)}$

$$\int \frac{\mathrm{d}^3k'}{2\omega(k')}\left\{\int_{-\infty}^{+\infty} \mathrm{d}T\, \Psi_T^{(X)}(\boldsymbol{k})\Psi_T^{(X)*}(\boldsymbol{k}')\right\}\Phi^{(X)}(\boldsymbol{k}') = \Phi^{(X)}(\boldsymbol{k}) \tag{39}$$

as should be expected. In addition, using the expressions (33) for Q^0 and (35) for Z, the following commutation rule is derived

$$[Q^0, Z] = -\mathrm{i}. \tag{40}$$

The spectral support of both Q^0 and Z is the whole real line, so that no difficulties arise from the Stone–Von Neumann theorem with (40) as would be the case were it to involve ω instead of Z. Finally, a comment on the relation between the time and the position operators is in order: the eigenstates of $\boldsymbol{Q}$ with eigenvalue $\boldsymbol{X}$ (5) belong to the detected subspace $\mathcal{H}_{\mathrm{KG}}^{(X)}$. However, it is not possible to determine simultaneously both the position (or the momentum) and the time-of-arrival due to the fact that the corresponding operators do not commute.

6. Interpretation

The results obtained so far indicate that the operator formalism associated with the time-of-arrival at a point works to fit the quantum mechanical rules. Accordingly, one can interpret it in a novel but standard way as was done on physical grounds in [1] for one space dimension. Here, we will show that the formalism provides the tools with which to build the quantum mechanical interpretation to be given to the time-of-arrival operator. In other words, that it provides the mathematical framework sufficient to define the time-of-arrival properties of the particle and associate with them definite probabilities. For definiteness, we assume that we are analysing the time-of-arrival at the point X. First we split the Hilbert space $\mathcal{H}$ of states into never detected $\mathcal{H}_{\mathrm{ND}}$ and detected subspaces $\mathcal{H}_{\mathrm{D}}$; obviously $\mathcal{H} = \mathcal{H}_D \oplus \mathcal{H}_{\mathrm{ND}}$. Also, from the discussion in section 4, we know that $\mathcal{H}_D = \mathcal{H}^{(X)}$. This will be the Hilbert space appropriate to the analysis. In $\mathcal{H}^{(X)}$ we have defined the (regularized) Hermitian operator $Q^0(\boldsymbol{X})$, whose spectrum is $T \in \mathcal{R}$, the set of observable times-of-arrival at the point X. Having solved the eigenvalue problem for $Q^0(\boldsymbol{X})$, we obtained a complete and orthogonal set of eigenfunctions $\psi_T^{(X)}(\boldsymbol{k}) = \langle \boldsymbol{k}|T, \boldsymbol{X}\rangle$ in the momentum representation. From them, we can define the set of elementary projectors $\{\Pi_T^{(X)}, T \in \mathcal{R}\}$ where

$$\Pi_T^{(X)} = |T, \boldsymbol{X}\rangle\langle T, \boldsymbol{X}|. \tag{41}$$

They generate a Boolean algebra $\mathcal{B}$ with the properties

$$\Pi_T^{(X)\dagger} = \Pi_T^{(X)} \qquad \Pi_T^{(X)}\Pi_{T'}^{(X)} = \delta(T - T')\Pi_T^{(X)}. \tag{42}$$

To each elementary projector there corresponds an event ($\Pi_T^{(X)} \leftrightarrow$ arrival at time T). Given any two projectors Π, $\Pi' \in \mathcal{B}$ the meet (*and*) and join (*or*) operations are defined as usual by

$$\Pi \wedge \Pi' = \Pi\Pi' \qquad \Pi \vee \Pi' = \Pi + \Pi' - \Pi\Pi' \tag{43}$$

where the notation corresponding to a finite-dimensional Boole algebra has been displayed for simplicity. Statements will, in general, be of the form $(Q^0(\boldsymbol{X}), T_1 < T < T_2)$, i.e. the particle arrives at X in the interval (T_1, T_2). Associated with them there will be projectors built by the joining of elementary projectors of the algebra

$$\Pi^{(X)}(T_1, T_2) = \int_{T_1}^{T_2} \mathrm{d}T\ \Pi_T^{(X)} \tag{44}$$

with matrix elements

$$\langle T, \boldsymbol{X}|\Pi^{(X)}(T_1, T_2)|T', \boldsymbol{X}\rangle = \delta(T - T')\theta(T_2 - T)\theta(T - T_1). \tag{45}$$

Finally, the algebra has to provide a decomposition of the identity suitable for the analysis of the properties of the observable under discussion, i.e.

$$\Pi^{(X)} \equiv \int_{-\infty}^{+\infty} \mathrm{d}T\ \Pi_T^{(X)} = 1 \tag{46}$$

which is valid in $\mathcal{H}^{(X)}$ due to (39), with the obvious meaning that an arbitrary state of $\mathcal{H}^{(X)}$ will not escape from detection. When acting on states belonging to Hilbert spaces larger than $\mathcal{H}^{(X)}$ the value of $\Pi^{(X)}$ will be smaller than one.

The complement of the statement $\Pi^{(X)}(T_1, T_2)$, i.e. the particle arrives at X at a time outside the interval (T_1, T_2) will be given by the projector $\Pi^{(X)} - \Pi^{(X)}(T_1, T_2)$. In the case that the state of the particle belongs to $\mathcal{H}^{(X)}$ the complement gives simply $1 - \Pi^{(X)}(T_1, T_2)$. The statement that there are states that escape from detection, absolutely when their projection on the detected subspace vanishes, or partially when they do not belong to $\mathcal{H}^{(X)}$ but have a finite projection on it, is given by the projector $1 - \Pi^{(X)}$. Finally, joining this last to the complement, gives the negative statement $1 - \Pi^{(X)}(T_1, T_2)$, i.e. the particle does not arrive at X in the interval (T_1, T_2). The fact that the negation and the complement may differ is a consequence of the incomplete character of the spectral decomposition of the time-of-arrival operator ($\Pi^{(X)} < 1$). This could be avoided by working inside $\mathcal{H}^{(X)}$ only, but this is too small to be of practical interest, consisting only of spherical waves about X.

We can now assign probabilities to the statements represented by the projectors of the algebra $\mathcal{B}$. Given an arbitrary normalized state Φ of the physical Hilbert space, the probability (in time) of arriving during the interval (T_1, T_2) at the position $\boldsymbol{X}$, $P_T^{(X)}(\Phi)$ is given by

$$P_{(T_1,T_2)}^{(X)}(\Phi) = \int_{T_1}^{T_2} \mathrm{d}T\ |\langle T, \boldsymbol{X}|\Phi\rangle|^2. \tag{47}$$

An arbitrary state Φ does not need to be in $\mathcal{H}_{\mathrm{D}}$, but in general will have a finite projection on it. Accordingly, we can define the probability of ever being detected at $\boldsymbol{X}$ by

$$P^{(X)}(\Phi) = \int_{-\infty}^{+\infty} \mathrm{d}T\ |\langle T, \boldsymbol{X}|\Phi\rangle|^2. \tag{48}$$

This will be equal to one for normalized states in $\mathcal{H}_{\mathrm{D}}$, as can be obtained from (39). For states not in $\mathcal{H}_{\mathrm{D}}$ this describes the case of states that classically would never be detected at the position $\boldsymbol{X}$, but quantum mechanically have a—less than one, but finite—probability for (ever) being detected at that point. Consider for example the ideal situation in which we place a detector along the *ox* axis at $\boldsymbol{X} = (x, 0, 0)$, and prepare at $t = 0$ a Gaussian wave packet centred at the origin, with mean momentum slightly off the *ox* axis $\langle \boldsymbol{k}\rangle = (k_0 \sin\theta, 0, k_0 \cos\theta)$. We consider the uncertainties in position and momentum to be such that wave packet and detector are well separated at $t = 0$, and the cone of flight of the particle ($\delta\theta \sim \Delta k/k$) misses the detector. Even in this case there will be a small probability for the particle ever being detected at $\boldsymbol{X}$; it is given by $P^{(X)}(\Phi)$. The probability of being detected during the interval (T_1, T_2) will be given by $P^{(X)}_{(T_1,T_2)}(\Phi)$, while the average value of the time-of-arrival operator will be

$$\langle Q^0(\boldsymbol{X})\rangle = \frac{\int_{-\infty}^{+\infty} \mathrm{d}T\, T\, |\langle T, \boldsymbol{X}|\Phi\rangle|^2}{\int_{-\infty}^{+\infty} \mathrm{d}T\, |\langle T, \boldsymbol{X}|\Phi\rangle|^2}. \tag{49}$$

This is a conditional average value, i.e. it makes sense only in the case when the particle is ever detected. Speaking about the value of the time-of-arrival in the other case is a logical contradiction, undefined mathematically, as in this case $\langle T, \boldsymbol{X}|\Phi\rangle = 0$.

The question of the time-of-arrival still deserves further clarification in quantum mechanics. We have outlined the mathematical framework whose existence allows for the assignment of probabilities to its different statements and for the use of logic to make inferences. In doing this, we are implicitly considering the existence of measurement devices (detectors in this case) which will function almost ideally, without introducing serious disturbances in the experimental results, so that the logical outcomes can be compared straightforwardly with the actual results. The existence of such detectors goes beyond the scope of the present work, which only deals with the formalism and its interpretation. This is a question common to this (distributions in time), and the usual (distributions in space) formulations of quantum mechanics. Another serious issue, of actual interest for its practical implications, is the inclusion of interactions in the formalism. For instance, how will the gravity field of the Earth modify the distribution of times-of-arrival as measured in the laboratory? This is of interest as there are experiments based on the production of a time-of-flight spectrum against the force of gravity. Another question is that of the time-of-arrival at a detector of a particle after traversing a barrier by quantum tunnelling. There is no classical analogue to this situation. Therefore the method presented here will be useless to address this problem, which calls for a completely quantum mechanical approach. There is a long list of pending questions worth further research. Here we turn to one of the motivations of this work, using the relativistic particle as a guideline to learn about time in quantum gravity. In principle, it would be plausible to think of the space part of the metric as playing a role similar to that of the detector position. Then, constraints restricting the detected Hilbert space as in (18) are likely to appear. Were this the case, the comparison would be among different possible initial states (of the Universe (?)), and the subject of comparison the time employed by these states to—or the probability of—'evolve' [12] to a definite space metric. All this is highly speculative and the object of further research. First, the mere existence of a suitable classical scheme from which to derive a time operator in the general case is not even clear.

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